

## James-Stein Estimator

### Situation

You want to estimate unknown parameter  $W$  of dimension  $p \geq 3$  from a single sample  $X$  such that the risk of the estimate is as low as possible.

You know the parameter  $W$  is drawn from a Gaussian distribution with mean 0 and variance  $\sigma^2$ :

$$W \sim N(0, \sigma^2 I).$$

The risk of the estimate is defined as the expected value of the loss.

The loss is defined as the L2 norm of the difference between the unknown parameter  $W$  and the estimate you provide  $e$ .

$$\text{Thus } \text{risk}(W, e) = E \text{ Loss}(W, e) = E \|W - e\|^2.$$

### An Incorrect Answer

Since you have just one sample  $X$ , use it as the estimate of  $W$ . To make things concrete, suppose that  $p = 3$  and  $X = [0.4, -0.05, 0.18]$ .

The risk is then

$$\text{risk}(W, X) = E \text{ Loss}(W, X) = E \|W - X\|^2 = E \|[w_1 - 0.4, w_2 + 0.05, w_3 - 0.18]\|^2.$$

### The Correct Answer

Instead of using  $X$ , use  $Y = (1 - (p - 2) \sigma^2 / \|X\|^2) X$ , which uses the L2 norm of  $X$ .

For example if  $\sigma^2 = 1$  (so that  $W \sim N(0, 1)$ ),  $p = 3$ , then using  $X$  as defined above,  $Y = (1 - (3 - 2) \cdot 1 / \|X\|^2) X = (1 - 1 / 0.44) X \approx (1 - 2.27) X = -1.27 X$ .

In James-61, a proof is provided that remarkably

$$\text{risk}(W, Y) < \text{risk}(W, X) \text{ for all } W, \text{ provided that } p \geq 3 \text{ and } W \sim N(0, \sigma^2 I).$$

Continuing the example, consider the point  $W = 0$ . Then

$$\text{risk}(0, X) = E \|[0.4, -0.05, 0.18]\|^2 \approx E(0.44) = 0.44$$

and

$$Y = (1 - 1 / \|X\|^2) X \approx (1 - 1 / 0.44) X = (1 - 2.27) X = -1.27 X$$

$$\text{risk}(0, Y) \approx E \|[(-1.27) [0.4, -0.05, 0.18] \|^2 \approx E(-1.27)(0.44) = E(-0.56) < E(0.44) = \text{risk}(0, X).$$

Another example. Suppose the same was  $X = [1,2,3]$  and again  $W = 0$ . Then  
 $\text{risk}(0,X) = E \|X\|^2 = E \| [1,2,3] \|^2 \approx E (3.87) = 3.87$   
and

$Y = (1 - 1 / \|X\|^2) X \approx (1 - 1 / 3.87) X = 0.74 X$   
 $\text{risk}(0,Y) \approx E \| 0.74 X \|^2 < E \|X\|^2$ .

These examples show that the norm of  $X$  must be sufficiently large or the multiplier becomes negative.

Ref: [James-61]. W. James and Charles Stein, "Estimation With Quadratic Loss", Proc. Fourth Berkeley Symp. on Math Statist and Prob., 1961, pp 361-379. Can be downloaded from <http://projecteuclid.org/DPubS?service=UI&version=1.0&verb=Display&handle=euclid.bsmsp/1200512173>